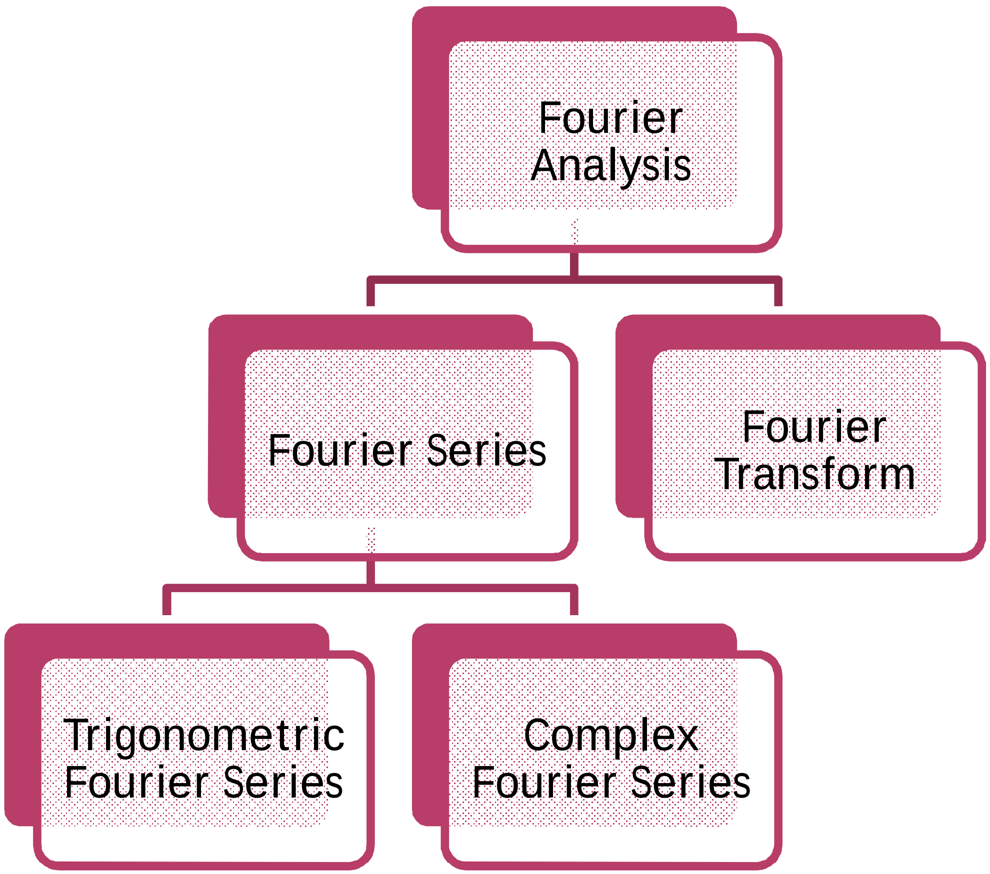
Fourier Series


Fourier Analysis
Fourier Series
Fourier Transform
Trigonometric Fourier Series
Complex Fourier Series

FOURIER ANALYSIS

Fourier analysis is a tool that changes a time domain signal to a frequency domain signal and vice versa

FOURIER SERIES

- Every composite periodic signal can be represented with a series of sine and cosine functions.
- The functions are integral harmonics of the fundamental frequency “f” of the composite signal.
- Using the series we can decompose any periodic signal into its harmonics

NEED OF FOURIER SERIES

- To convert a signal into sinusoidal , we require a mathematical formula .
- Fourier series provide such a tool, which can convert a signal into sinusoidal.

DIRICHLET CONDITIONS

- A periodic signal $x(t)$, has a Fourier series if it satisfies the following conditions:

1. $x(t)$ is absolutely integrable over any period, namely

$$\int_{a}^{a+T} |x(t)|\,dt < \infty, \quad \forall a \in \mathbb{R}$$

2. $x(t)$ has only a finite number of maxima and minima over any period
3. $x(t)$ has only a finite number of discontinuities over any period

TRIGNOMETRIC FOURIER SERIES

$$g(t) = a_0 + a_1 \; cos \; \omega_0 t + a_2 \; cos \; 2\omega_0 t + a_3 \; cos \; 3\omega_0 t + \ldots\ldots\ldots\ldots + a_{(n-1)} \; cos \; (n-1)\omega_0 t + a_n \; cos \; n\omega_0 t + b_1 \; sin \; \omega_0 t + b_2 \; sin \; 2\omega_0 t + b_3 \; sin \; 3\omega_0 t + \ldots\ldots\ldots\ldots + b_n \; sin \; n\omega_0 t$$

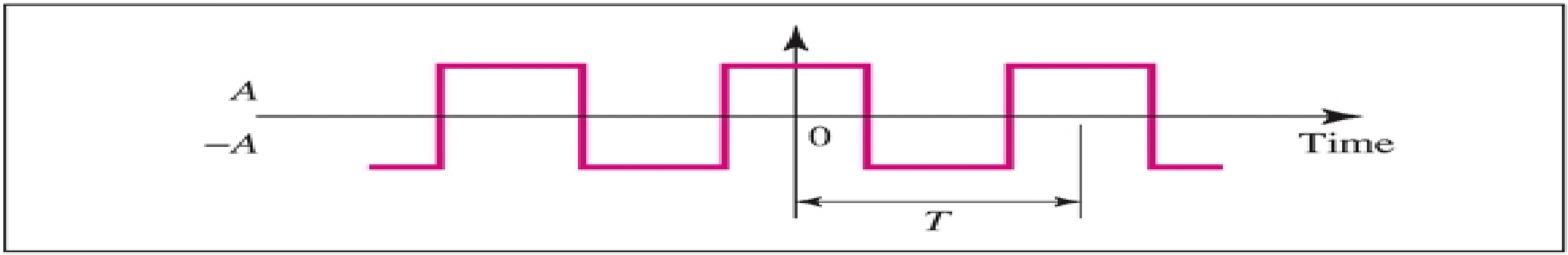
$$a_n = \frac{2}{T}\int_0^T g(t) \; cos \; n\omega_0 t \; dt$$

$$b_n = \frac{2}{T}\int_0^T g(t) \; sin \; n\omega_0 t \; dt$$

$$a_0 = \frac{1}{T}\int_0^T g(t) \; dt$$

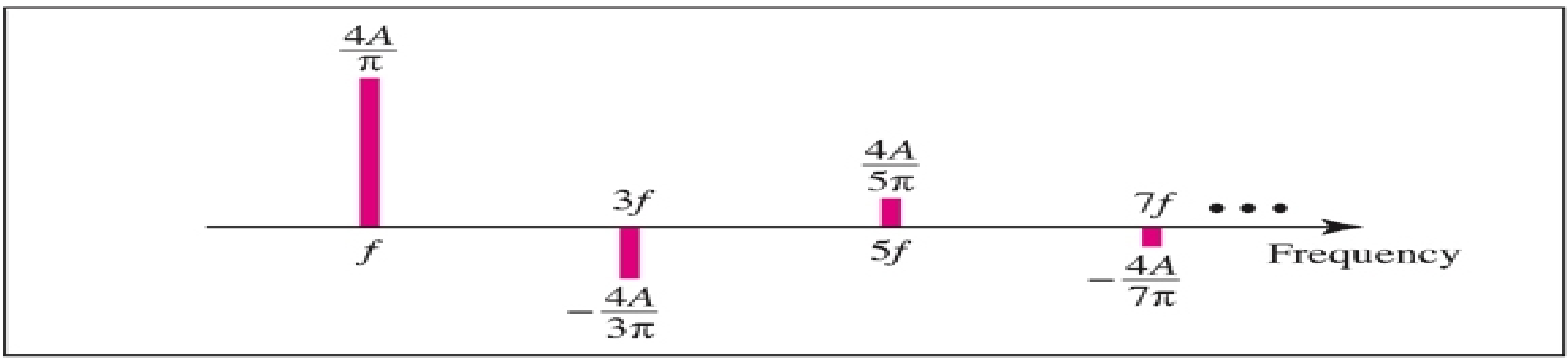
EXAMPLES OF SIGNALS AND THE FOURIER SERIES REPRESENTATION

Time domain



$$A_0 = 0 \qquad A_n = \begin{cases} \dfrac{4A}{n\pi} & \text{for } n = 1, 5, 9, \ldots \\ -\dfrac{4A}{n\pi} & \text{for } n = 3, 7, 11, \ldots \end{cases} \qquad B_n = 0$$

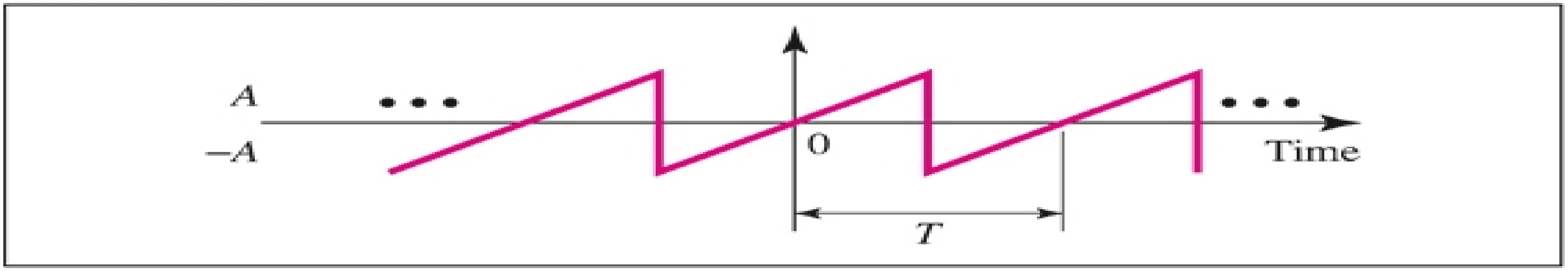
$$s(t) = \frac{4A}{\pi}\cos(2\pi ft) - \frac{4A}{3\pi}\cos(2\pi 3ft) + \frac{4A}{5\pi}\cos(2\pi 5ft) - \frac{4A}{7\pi}\cos(2\pi 7ft) + \cdots$$



Frequency domain

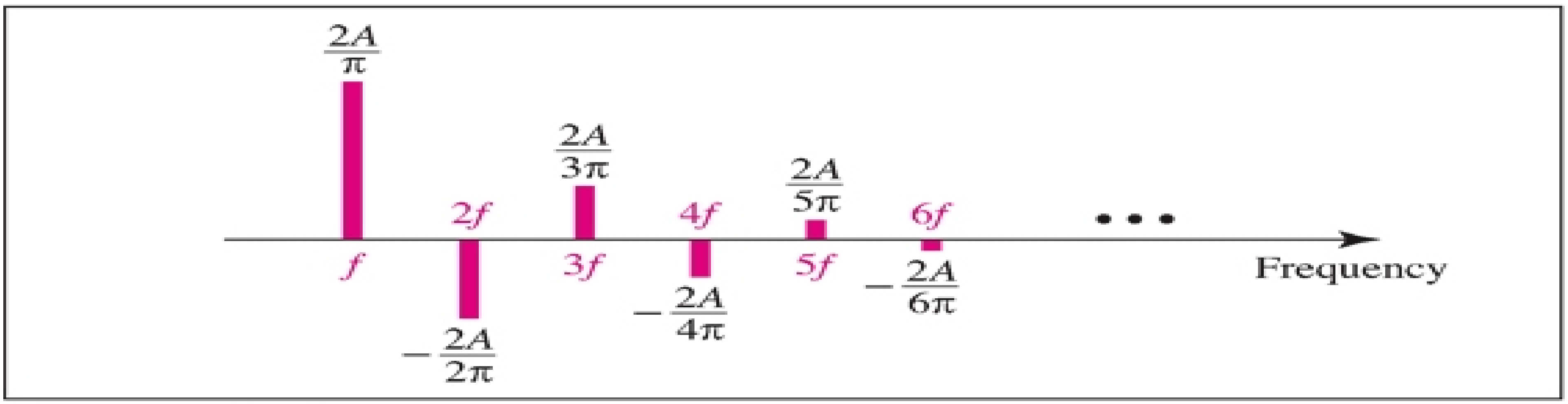
SAWTOOTH SIGNAL

Time domain



$A_0 = 0$ $\quad A_n = 0$ $\quad B_n = \begin{cases} \dfrac{2A}{n\pi} & \text{for } n \text{ odd} \\ -\dfrac{2A}{n\pi} & \text{for } n \text{ even} \end{cases}$

$$s(t) = \frac{2A}{\pi}\sin(2\pi f t) - \frac{2A}{2\pi}\sin(2\pi 2 f t) + \frac{2A}{3\pi}\sin(2\pi 3 f t) - \frac{2A}{4\pi}\sin(2\pi 4 f t) + \cdots$$



Frequency domain

COMPLEX FOURIER SERIES

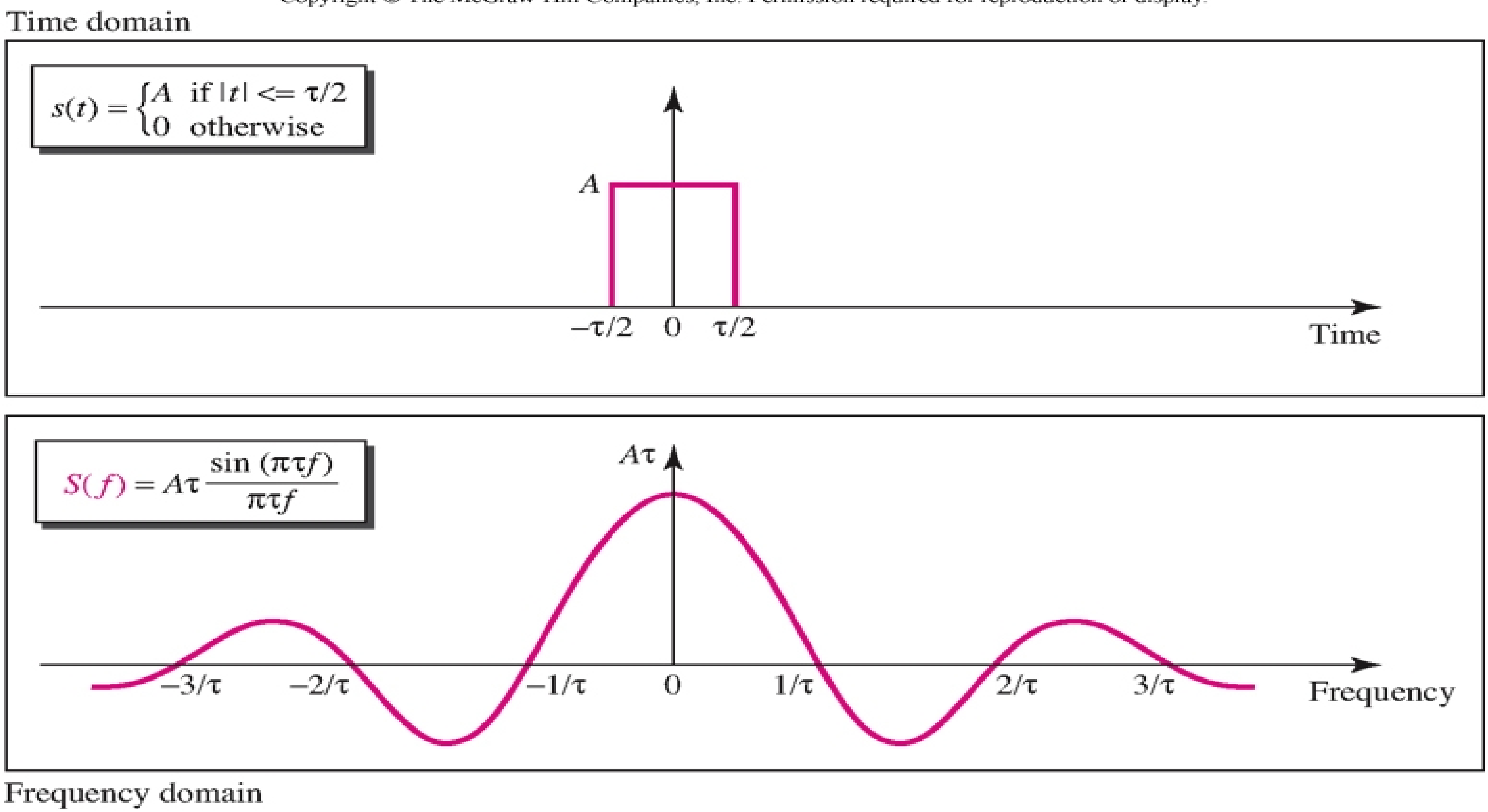
$$g(t) = \sum_{n=-\infty}^{+\infty} C_n \; e^{jn\omega_0 t}$$

$$C_n = \frac{1}{T} \int_0^T g(t) \; e^{-jn\omega_0 t} \; dt$$

FOURIER TRANSFORM

- Fourier Transform gives the frequency domain of a nonperiodic time domain signal

EXAMPLE OF A FOURIER TRANSFORM



FOURIER TRANSFORM

$$F[g(t)] = G(f) = \int_{-\infty}^{+\infty} g(t)\ e^{-j\omega t}\ dt$$

$$F^{-1}[G(f)] = g(t) = \int_{-\infty}^{+\infty} G(f)\ e^{+j\omega t}\ df$$

PROPERTIES OF FT

Operation	Time Function	Fourier Transform
Linearity	$af_1(t) + bf_2(t)$	$aF_1(\omega) + bF_2(\omega)$
Time shift	$f(t - t_0)$	$F(\omega)e^{-j\omega t_0}$
Time scaling	$f(at)$	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)$
Time transformation	$f(at - t_0)$	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)e^{-j\omega t_0/a}$
Duality	$F(t)$	$2\pi f(-\omega)$
Frequency shift	$f(t)e^{j\omega_0 t}$	$F(\omega-\omega_0)$
Convolution	$f_1(t)*f_2(t)$	$F_1(\omega)F_2(\omega)$
	$f_1(t)f_2(t)$	$\frac{1}{2\pi}F_1(\omega)*F_2(\omega)$
Differentiation	$\frac{d^n[f(t)]}{dt^n}$	$(j\omega)^n F(\omega)$
	$(-jt)^n f(t)$	$\frac{d^n[F(\omega)]}{d\omega^n}$
Integration	$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{1}{j\omega}F(\omega) + \pi F(0)\delta(\omega)$

TIME LIMITED AND BAND LIMITED SIGNALS

- A time limited signal is a signal for which the amplitude $s(t) = 0$ for $t > T_1$ and $t < T_2$
- A band limited signal is a signal for which the amplitude $S(f) = 0$ for $f > F_1$ and $f < F_2$

PARSEVAL'S ENERGY THEOREM

- Mathematical technique to find out the energy of a signal in frequency domain by using Fourier transform.
- When we know the Fourier transform of signal, its energy can be calculated without converting into time domain.

$$E = \int_{-\infty}^{+\infty} |G(f)|^2 \, df$$

- It is also called Rayleigh's energy theorem.

- Proof: Energy of a signal in time domain

$$E= \int_{-\infty}^{+\infty} |g(t)|^2 \; dt$$

$$E= \int_{-\infty}^{+\infty} |g(t).g(t)| \; dt$$

- Inverse FT

$$g(t)= \int_{-\infty}^{+\infty} G(f) \; e^{j\omega t} \; df$$

- By putting g(t)

$$E= \int_{-\infty}^{+\infty} |g(t)| \left\{ \int_{-\infty}^{+\infty} G(f) e^{j\omega t} df \right\} \; dt$$

- By interchanging the order of integration

$$E= \int_{-\infty}^{+\infty} |G(f)| \ df \int_{-\infty}^{+\infty} g(t)e^{j\omega t}\, dt$$

- By the concept of complex conjugate

$$G^{*}(f) \ =G(-f) \ = \ \int_{-\infty}^{+\infty} g(t) \ e^{j\omega t} \ dt$$

- Where $G^{*}(f)$ is complex conjugate of G(f), so by putting

$$E= \int_{-\infty}^{+\infty} |G(f).G^{*}(f)| \ df$$

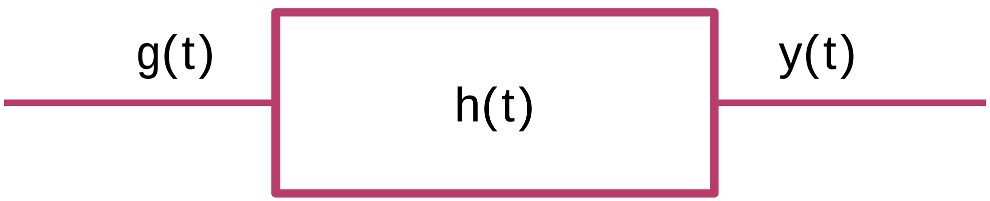
$$E= \int_{-\infty}^{+\infty} |G(f)|^{2} \ df$$

ENERGY SPECTRAL DENSITY

- Defined as energy per unit bandwidth

$$ESD = |G(f)|^2$$

- Let signal g(t) is passed with a low pass filter



$$y(t) = g(t) * h(t)$$

- Taking FT

$$Y(f) = G(f) \,.\, H(f)$$

- FT of LPF lies between $-f_m$ to $+f_m$ with amplitude one

$$E = \int_{-\infty}^{+\infty} |Y(f)|^2 \, df$$

$$E = \int_{-\infty}^{+\infty} |G(f).H(f)|^2 \, df$$

$$E = \int_{-f_m}^{+f_m} |G(f)|^2 \, df$$

$$E = |G(f)|^2 \int_{-f_m}^{+f_m} df$$

$$E = |G(f)|^2 . \; 2f_m$$

$$\frac{E}{2f_m} = |G(f)|^2 \Longrightarrow \frac{E}{B} = |G(f)|^2$$

$$ESD = |G(f)|^2$$

POWER SPECTRAL DENSITY

- Defined as power per unit bandwidth.
- Let the g(t) is defined as

$$g(t) = \begin{cases} g(t) & -\frac{T}{2} \leq t \leq +\frac{T}{2} \\ 0 & otherwise \end{cases}$$

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} |g(t)|^2 \, dt$$

$$P = \lim_{T\to\infty} \frac{1}{T} \left[\int_{-\infty}^{-T/2} |g(t)|^2 \, dt + \int_{-T/2}^{-0} |g(t)|^2 \, dt \right]$$

$$+ \lim_{T\to\infty} \frac{1}{T} \left[\int_{0}^{+T/2} |g(t)|^2 \, dt + \int_{+T/2}^{\infty} |g(t)|^2 \, dt \right]$$

- But we know g(t) is defined for only –T/2 to +T/2
- So the power content between -∞ to –T/2 and +T/2 to ∞ is zero.

$$\lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{+\infty} |g(t)|^2 \ dt$$

- By Parseval energy theorem

$$\lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{+\infty} |G(f)|^2 \ df$$

$$P = \lim_{T\to\infty} \frac{1}{T} |G(f)|^2 \int_{-\infty}^{+\infty} \ df$$

- But if g(t) is defined between –T/2<= t<= +T/2 Then G(f) must be lies in the range of $+f_m$ to f_m

$$\frac{P}{\int_{-\infty}^{+\infty} df} = \lim_{T \to \infty} \frac{1}{T} |G(f)|^2$$

$$PSD[S(f)] = \lim_{T \to \infty} \frac{|G(f)|^2}{T}$$